

## Gaussian noises case

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As we know:

$$\Delta\hat{\mathbf{q}} = N_1\Delta\check{\mathbf{q}} + N_2\mathbf{q} \quad (1)$$

$$E\{\Delta\hat{\mathbf{q}}\} = \bar{N}_2\mathbf{q} \quad (2)$$

$$\Delta\hat{\mathbf{q}} - E\{\Delta\hat{\mathbf{q}}\} = N_1\Delta\check{\mathbf{q}} + (N_2 - \bar{N}_2)\mathbf{q} \quad (3)$$

where

$$N_1 = I_4 - \mathbf{q}\mathbf{q}^T \quad (4)$$

$$N_2 = \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T + \frac{1}{2} \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4 \quad (5)$$

$$\bar{N}_2 = P_{\Delta\check{\mathbf{q}}} + \frac{1}{2} \text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \quad (6)$$

$$M = I_4 - 3\mathbf{q}\mathbf{q}^T \quad (7)$$

Then

$$P_{\Delta\hat{\mathbf{q}}} = E\{(N_1\Delta\check{\mathbf{q}} + (N_2 - \bar{N}_2)\mathbf{q})(N_1\Delta\check{\mathbf{q}} + (N_2 - \bar{N}_2)\mathbf{q})^T\} \quad (8)$$

$$= N_1 P_{\Delta\check{\mathbf{q}}} N_1^T + E\{(N_2 - \bar{N}_2)\mathbf{q}\mathbf{q}^T(N_2 - \bar{N}_2)^T\} \quad (9)$$

$$= N_1 P_{\Delta\check{\mathbf{q}}} N_1^T + P_{\Delta\check{\mathbf{q}}} \mathbf{q}\mathbf{q}^T P_{\Delta\check{\mathbf{q}}} + \frac{1}{2} P_{\Delta\check{\mathbf{q}}} \mathbf{q}\mathbf{q}^T \text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \quad (10)$$

$$+ \frac{1}{2} \text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \mathbf{q}\mathbf{q}^T P_{\Delta\check{\mathbf{q}}} + \frac{1}{4} \text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \mathbf{q}\mathbf{q}^T \text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \quad (11)$$

$$- E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T P_{\Delta\check{\mathbf{q}}}\} - \frac{1}{2} E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4\} \quad (12)$$

$$- \frac{1}{2} E\{[\Delta\check{\mathbf{q}}^T M \Delta\check{\mathbf{q}} I_4] \mathbf{q}\mathbf{q}^T P_{\Delta\check{\mathbf{q}}}\} - \frac{1}{4} E\{[\Delta\check{\mathbf{q}}^T M \Delta\check{\mathbf{q}} I_4] \mathbf{q}\mathbf{q}^T \text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4\} \quad (13)$$

$$- E\{P_{\Delta\check{\mathbf{q}}} \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} - \frac{1}{2} E\{P_{\Delta\check{\mathbf{q}}} \mathbf{q}\mathbf{q}^T [\Delta\check{\mathbf{q}}^T M \Delta\check{\mathbf{q}} I_4]\} \quad (14)$$

$$- \frac{1}{2} E\{\text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\} - \frac{1}{4} E\{\text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \Delta\check{\mathbf{q}}^T \mathbf{q} \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4\} \quad (15)$$

$$+ E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{2} E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4\} \quad (16)$$

$$+ \frac{1}{2} E\{\text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4 \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T + \frac{1}{4} E\{\text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4 \mathbf{q}\mathbf{q}^T \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4\} \quad (17)$$

$$= N_1 P_{\Delta\check{\mathbf{q}}} N_1^T - E\{P_{\Delta\check{\mathbf{q}}} \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} - \frac{1}{2} E\{P_{\Delta\check{\mathbf{q}}} \mathbf{q}\mathbf{q}^T [\Delta\check{\mathbf{q}}^T M \Delta\check{\mathbf{q}} I_4]\} \quad (18)$$

$$- \frac{1}{2} E\{\text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\} - \frac{1}{4} E\{\text{tr}(MP_{\Delta\check{\mathbf{q}}}) I_4 \Delta\check{\mathbf{q}}^T \mathbf{q} \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4\} \quad (19)$$

$$+ E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{2} E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4\} \quad (20)$$

$$+ \frac{1}{2} E\{\text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4 \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T + \frac{1}{4} E\{\text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4 \mathbf{q}\mathbf{q}^T \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4\} \quad (21)$$

The fourth-order terms in  $P_{\Delta\check{\mathbf{q}}}$  is:

$$P_{\Delta\check{\mathbf{q}}}^{4th} = + E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{2}E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4\} \quad (22)$$

$$+ \frac{1}{2}E\{\text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4 \mathbf{q}\mathbf{q}^T \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T + \frac{1}{4}E\{\text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4 \mathbf{q}\mathbf{q}^T \text{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T) I_4\} \quad (23)$$

$$= + E\{(\Delta\check{\mathbf{q}}^T \mathbf{q})^2 \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{2}E\{(\Delta\check{\mathbf{q}}^T \Delta\check{\mathbf{q}})\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T\} + \frac{1}{2}E\{(\Delta\check{\mathbf{q}}^T \Delta\check{\mathbf{q}})\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T \mathbf{q}\mathbf{q}^T\}^T \quad (24)$$

$$- \frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T \mathbf{q})^3 \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} - \frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T \mathbf{q})^3 \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\}^T + \frac{1}{4}E\{(\Delta\check{\mathbf{q}}^T \Delta\check{\mathbf{q}})^2 \mathbf{q}\mathbf{q}^T\} \quad (25)$$

$$- \frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T \mathbf{q})^2 (\Delta\check{\mathbf{q}}^T \Delta\check{\mathbf{q}}) \mathbf{q}\mathbf{q}^T\} + \frac{9}{4}E\{(\Delta\check{\mathbf{q}}^T \mathbf{q})^4 \mathbf{q}\mathbf{q}^T\} \quad (26)$$

In terms of the fourth-order we use Guassian noise to  $\Delta\check{\mathbf{q}}$ : Let  $\mathbf{q} = [q_1, q_2, q_3, q_4]^T$ , and  $\Delta\check{\mathbf{q}} = [x_1, x_2, x_3, x_4]^T$ ,  $x_i \sim N(0, \sigma_{ij})$ ,

$$E\{x_i x_j x_k x_m\} = E\{x_i x_j\}E\{x_k x_m\} + E\{x_i x_k\}E\{x_j x_m\} + E\{x_i x_m\}E\{x_j x_k\} \quad (27)$$

$$+ E\{x_i\}E\{x_j\}E\{x_k\}E\{x_m\} \quad (28)$$

$$E\{x_i x_j x_k x_m\} = \sigma_{ij}\sigma_{km} + \sigma_{ik}\sigma_{jm} + \sigma_{im}\sigma_{jk} + \sigma_i^2 \sigma_j^2 \sigma_k^2 \sigma_m^2 \quad (29)$$

$$E\{x_i^2 x_j x_k\} = \sigma_i^2 \sigma_{jk} + 2\sigma_{ij}\sigma_{ik} \quad (30)$$

$$E\{x_i^2 x_j^2\} = \sigma_i^2 \sigma_j^2 + 2\sigma_{ij}^2 \quad (31)$$

$$E\{x_i^3 x_j\} = 3\sigma_i^2 \sigma_{ij} \quad (32)$$

$$E\{x_i^4\} = 3\sigma_i^4 \quad (33)$$

There is the expectation matrix for each term:

$$1. E\{(\Delta\check{\mathbf{q}}^T \mathbf{q})^2 \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} = E \left( \begin{array}{cccc} q_1^2 x_1^4 + q_2^2 x_1^2 x_2^2 & q_2^2 x_1 x_2^3 + 2q_1 q_2 x_1^2 x_2^2 & q_3^2 x_1 x_2^3 + 2q_1 q_3 x_1^2 x_2^2 & q_4^2 x_1 x_2^3 + 2q_1 q_4 x_1^2 x_2^2 \\ + 2q_1 q_2 x_1^3 x_2 + q_3^2 x_1^2 x_2^2 & + q_1^2 x_1^3 x_2 + q_3^2 x_1 x_2 x_2^2 & + 2q_2 q_3 x_1 x_2 x_2^2 + q_1^2 x_1^3 x_2 & + 2q_2 q_4 x_1 x_2 x_2^2 + 2q_3 q_4 x_1 x_2 x_2^2 \\ + 2q_1 q_3 x_1^3 x_2 + 2q_2 q_3 x_1^2 x_2 x_3 & + 2q_2 q_3 x_1 x_2^2 x_3 + 2q_1 q_3 x_1^2 x_2 x_3 & + q_2^2 x_1 x_2^2 x_3 + 2q_1 q_2 x_1^2 x_2 x_3 & + q_1^2 x_1^3 x_2 + q_2^2 x_1 x_2^2 x_3 \\ + q_4^2 x_1^2 x_2^2 + 2q_1 q_4 x_1^3 x_2 & + q_4^2 x_1 x_2 x_2^2 + 2q_2 q_4 x_1 x_2^2 x_3 & + q_4^2 x_1 x_2 x_2^2 + 2q_3 q_4 x_1 x_2^2 x_3 & + 2q_1 q_2 x_1^2 x_2 x_3 + q_3^2 x_1 x_2^2 x_3 \\ + 2q_2 q_4 x_1^2 x_2 x_3 + 2q_3 q_4 x_1^2 x_2 x_3 & + 2q_1 q_4 x_1^2 x_2 x_3 + 2q_3 q_4 x_1 x_2 x_2 x_3 & + 2q_1 q_4 x_1^2 x_2 x_3 + 2q_2 q_4 x_1 x_2 x_2 x_3 & + 2q_1 q_3 x_1 x_2 x_2 x_3 + 2q_2 q_3 x_1 x_2 x_2 x_3 \\ q_2^2 x_1 x_2^2 + 2q_1 q_2 x_1^2 x_2^2 & q_2^2 x_2^4 + 2q_1 q_2 x_1 x_2^3 & q_3^2 x_2 x_2^3 + 2q_2 q_3 x_2^2 x_2^2 & q_3^2 x_2 x_2^3 + 2q_2 q_3 x_2^2 x_2^2 \\ + q_1^2 x_1^3 x_2 + q_3^2 x_1 x_2 x_2^2 & + q_1^2 x_1^2 x_2^2 + q_3^2 x_2^2 x_2^2 & + 2q_1 q_3 x_1 x_2 x_2^2 + q_2^2 x_2^3 x_2^2 & + 2q_1 q_3 x_1 x_2 x_2^2 + q_2^2 x_2^3 x_2^2 \\ + 2q_2 q_3 x_1 x_2^2 x_3 + 2q_1 q_3 x_1^2 x_2 x_3 & + 2q_2 q_3 x_2^3 x_3 + 2q_1 q_3 x_1 x_2^2 x_3 & + 2q_1 q_2 x_1 x_2^2 x_3 + q_1^2 x_1^2 x_2 x_3 & + 2q_1 q_2 x_1 x_2^2 x_3 + q_1^2 x_1^2 x_2 x_3 \\ + q_4^2 x_1 x_2 x_2^2 + 2q_2 q_4 x_1 x_2^2 x_3 & + q_4^2 x_2^2 x_2^2 + 2q_2 q_4 x_2^2 x_3 & + q_4^2 x_2 x_2 x_2^2 + 2q_3 q_4 x_2 x_2^2 x_3 & + q_4^2 x_2 x_2 x_2^2 + 2q_3 q_4 x_2 x_2^2 x_3 \\ + 2q_1 q_4 x_1^2 x_2 x_3 + 2q_3 q_4 x_1 x_2 x_2 x_3 & + 2q_1 q_4 x_1 x_2^2 x_3 + 2q_3 q_4 x_2^2 x_2 x_3 & + 2q_2 q_4 x_2^2 x_3 x_3 + 2q_1 q_4 x_1 x_2 x_2 x_3 & + 2q_2 q_4 x_2^2 x_3 x_3 + 2q_1 q_4 x_1 x_2 x_2 x_3 \\ q_3^2 x_1 x_2^3 + 2q_1 q_3 x_1^2 x_2^2 & q_3^2 x_2 x_2^3 + 2q_2 q_3 x_2^2 x_2^2 & q_3^2 x_3^4 + 2q_1 q_3 x_1 x_2^3 & q_3^2 x_3^4 + 2q_1 q_3 x_1 x_2^3 \\ + 2q_2 q_3 x_1 x_2 x_2^2 + q_1^2 x_1^3 x_2 & + 2q_1 q_3 x_1 x_2 x_2^2 + q_2^2 x_2^3 x_2^2 & + 2q_2 q_3 x_2 x_2^2 + q_1^2 x_1^2 x_2^2 & + 2q_1 q_3 x_1 x_2 x_2^2 + q_2^2 x_2^3 x_2^2 \\ + q_2^2 x_1 x_2^2 x_3 + 2q_1 q_2 x_1^2 x_2 x_3 & + 2q_1 q_2 x_1 x_2^2 x_3 + q_1^2 x_1^2 x_2 x_3 & + q_2^2 x_2^2 x_2^2 + 2q_1 q_2 x_1 x_2 x_2^2 & + q_2^2 x_2^2 x_2^2 + 2q_1 q_2 x_1 x_2 x_2^2 \\ + q_4^2 x_1 x_2 x_2^2 + 2q_3 q_4 x_1 x_2^2 x_3 & + q_4^2 x_2 x_2 x_2^2 + 2q_3 q_4 x_2 x_2^2 x_3 & + q_4^2 x_3^2 x_2^2 + 2q_3 q_4 x_3 x_2^2 x_3 & + q_4^2 x_3^2 x_2^2 + 2q_3 q_4 x_3 x_2^2 x_3 \\ + 2q_1 q_4 x_1^2 x_2 x_3 + 2q_2 q_4 x_1 x_2 x_2 x_3 & + 2q_2 q_4 x_2^2 x_2 x_3 + 2q_1 q_4 x_1 x_2 x_2 x_3 & + 2q_1 q_4 x_1 x_2^2 x_3 + 2q_2 q_4 x_2 x_2^2 x_3 & + 2q_1 q_4 x_1 x_2^2 x_3 + 2q_2 q_4 x_2 x_2^2 x_3 \\ q_4^2 x_1 x_2^3 + 2q_1 q_4 x_1^2 x_2^2 & q_4^2 x_2 x_2^3 + 2q_2 q_4 x_2^2 x_2^2 & q_4^2 x_3 x_2^3 + 2q_3 q_4 x_3 x_2^2 x_2 & q_4^2 x_3 x_2^3 + 2q_3 q_4 x_3 x_2^2 x_2 \\ + 2q_2 q_4 x_1 x_2 x_2^2 + 2q_3 q_4 x_1 x_2^2 x_3 & + 2q_1 q_4 x_1 x_2 x_2^2 + 2q_3 q_4 x_2 x_2^2 x_3 & + 2q_1 q_4 x_1 x_2 x_2^2 + 2q_3 q_4 x_2 x_2^2 x_3 & + 2q_1 q_4 x_1 x_2 x_2^2 + 2q_3 q_4 x_2 x_2^2 x_3 \\ + q_1^2 x_1^3 x_2 + q_2^2 x_1 x_2^2 x_3 & + q_2^2 x_2^3 x_2 + 2q_1 q_2 x_1 x_2^2 x_3 & + q_3^2 x_3^2 x_2 + 2q_1 q_3 x_1 x_2^2 x_3 & + q_3^2 x_3^2 x_2 + 2q_1 q_3 x_1 x_2^2 x_3 \\ + 2q_1 q_2 x_1^2 x_2 x_3 + q_3^2 x_1 x_2^2 x_3 & + q_1^2 x_1^2 x_2 x_3 + q_3^2 x_2^2 x_2 x_3 & + 2q_2 q_3 x_2 x_2^2 x_3 + q_1^2 x_1^2 x_2 x_3 & + 2q_2 q_3 x_2 x_2^2 x_3 + q_1^2 x_1^2 x_2 x_3 \\ + 2q_1 q_3 x_1^2 x_2 x_3 + 2q_2 q_3 x_1 x_2 x_2 x_3 & + 2q_2 q_3 x_2^2 x_2 x_3 + 2q_1 q_3 x_1 x_2 x_2 x_3 & + q_2^2 x_2^2 x_2 x_3 + 2q_1 q_2 x_1 x_2 x_2 x_3 & + q_2^2 x_2^2 x_2 x_3 + 2q_1 q_2 x_1 x_2 x_2 x_3 \end{array} \right)$$

$$= \left( \begin{array}{lll} 3q_1^2\sigma_1^4 + q_2^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) & 3q_1^2\sigma_1^2\sigma_{12} + 3q_2^2\sigma_2^2\sigma_{12} & 3q_1^2\sigma_1^2\sigma_{13} + 3q_2^2\sigma_2^2\sigma_{13} & 3q_1^2\sigma_1^2\sigma_{14} + 3q_2^2\sigma_2^2\sigma_{14} \\ + 6q_1q_2\sigma_1^2\sigma_{12} + q_3^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) & + 2q_1q_2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) & + q_2^2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) & + q_2^2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 6q_1q_3\sigma_1^2\sigma_{13} + 6q_1q_4\sigma_1^2\sigma_{14} & + q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) & + 2q_1q_2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) & + 2q_1q_2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}^2) & + 2q_1q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) & + 2q_1q_3(\sigma_1^2\sigma_{23}^2 + 2\sigma_{13}^2) & + 2q_1q_4(\sigma_1^2\sigma_{24}^2 + 2\sigma_{14}^2) \\ + 2q_2q_3(\sigma_2^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) & + 2q_2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) & + 2q_2q_3(\sigma_{12}\sigma_3^2 + 2\sigma_{13}\sigma_{23}) & + 2q_2q_4(\sigma_{12}\sigma_4^2 + 2\sigma_{14}\sigma_{24}) \\ + 2q_2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) & + q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) & + q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) & + q_3^2(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ + 2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) & + 2q_1q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) & + 2q_1q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) & + 2q_1q_3(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\ & + 2q_2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) & + 2q_3q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{14}) & + 2q_3q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ & + 2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) & + 2q_2q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) & + 2q_2q_3(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \\ 3q_1^2\sigma_1^2\sigma_{12} + 3q_2^2\sigma_2^2\sigma_{12} & 3q_2^2\sigma_2^4 + q_1^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) & q_1^2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) & q_1^2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + 2q_1q_2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) & + 6q_1q_2\sigma_2^2\sigma_{12} + 6q_2q_4\sigma_2^2\sigma_{24} & + 3q_2^2\sigma_2^2\sigma_{23} + 3q_3^2\sigma_3^2\sigma_{32} & + 3q_2^2\sigma_2^2\sigma_{23} + 3q_4^2\sigma_4^2\sigma_{24} \\ + q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) & + q_3^2(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) & + 2q_1q_2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) & + 2q_1q_2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 2q_1q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) & + 6q_2q_3\sigma_2^2\sigma_{23} & + 2q_1q_3(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) & + 2q_1q_4(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\ + 2q_2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) & + 2q_1q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) & + 2q_2q_3(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) & + 2q_2q_4(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\ + q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) & + q_4^2(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) & + q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) & + q_3^2(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\ + 2q_1q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) & + 2q_1q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) & + 2q_2q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) & + 2q_2q_3(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\ + 2q_2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) & + 2q_3q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) & + 2q_3q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) & + 2q_3q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) & + 2q_1q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) & + 2q_1q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) & + 2q_1q_3(\sigma_{12}\sigma_{34} + \sigma_{14}\sigma_{23} + \sigma_{13}\sigma_{24}) \\ 3q_1^2\sigma_1^2\sigma_{13} + 3q_2^2\sigma_2^2\sigma_{13} & q_1^2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) & 3q_3^2\sigma_3^4 + 6q_2q_3\sigma_3^2\sigma_{23} & q_1^2(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\ + q_2^2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) & + 3q_2^2\sigma_2^2\sigma_{23} + 3q_3^2\sigma_3^2\sigma_{32} & + q_2^2(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) & + 3q_3^2\sigma_3^2\sigma_{34} + 3q_4^2\sigma_4^2\sigma_{43} \\ + 2q_1q_2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) & + 2q_1q_2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) & + q_1^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) & + 2q_1q_3(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ + 2q_1q_3(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) & + 2q_1q_3(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) & + 6q_1q_3\sigma_3^2\sigma_{13} + 6q_3q_4\sigma_3^2\sigma_{34} & + 2q_2q_3(\sigma_3^2\sigma_{42} + 2\sigma_{23}\sigma_{34}) \\ + 2q_2q_3(\sigma_{12}\sigma_3^2 + 2\sigma_{13}\sigma_{23}) & + 2q_2q_3(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) & + 2q_1q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) & + 2q_3q_4(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\ + q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{24}) & + q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) & + q_4^2(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) & + q_2^2(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\ + 2q_1q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) & + 2q_2q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) & + 2q_2q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) & + 2q_2q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_3q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{14}) & + 2q_3q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) & + 2q_2q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) & + 2q_1q_4(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\ + 2q_2q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) & + 2q_1q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) & + 2q_1q_2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) & + 2q_1q_2(\sigma_{12}\sigma_{34} + \sigma_{14}\sigma_{23} + \sigma_{13}\sigma_{24}) \\ 3q_1^2\sigma_1^2\sigma_{14} + 3q_2^2\sigma_2^2\sigma$$

$$2. \frac{1}{2} E\{(\Delta \tilde{\mathbf{q}}^T \Delta \tilde{\mathbf{q}}) \Delta \tilde{\mathbf{q}} \Delta \tilde{\mathbf{q}}^T \mathbf{q} \mathbf{q}^T\}:$$

$$\frac{1}{2} E \left( \begin{array}{cccc} x_1^4 + x_1^2 x_2^2 & x_1 x_2^3 + x_1^3 x_2 & x_1 x_3^3 + x_1^3 x_3 & x_1 x_4^3 + x_1^3 x_4 \\ + x_1^2 x_3^2 + x_1^2 x_4^2 & + x_1 x_2 x_3^2 + x_1 x_2 x_4^2 & + x_1 x_2^2 x_3 + x_1 x_3 x_4^2 & + x_1 x_2^2 x_4 + x_1 x_3^2 x_4 \\ x_1 x_2^3 + x_1^3 x_2 & x_2^4 + x_1^2 x_2^2 & x_2 x_3^3 + x_2^3 x_3 & x_2 x_4^3 + x_2^3 x_4 \\ + x_1 x_2 x_3^2 + x_1 x_2 x_4^2 & + x_2^2 x_3^2 + x_2^2 x_4^2 & + x_1^2 x_2 x_3 + x_2 x_3 x_4^2 & + x_1^2 x_2 x_4 + x_2 x_3^2 x_4 \\ x_1 x_3^3 + x_1^3 x_3 & x_2 x_3^3 + x_2^3 x_3 & x_3^4 + x_1^2 x_3^2 & x_3 x_4^3 + x_3^3 x_4 \\ + x_1 x_2^2 x_3 + x_1 x_3 x_4^2 & + x_2^2 x_2 x_3 + x_2 x_3 x_4^2 & + x_2^2 x_3^2 + x_2^2 x_4^2 & + x_2^2 x_3 x_4 + x_2^2 x_3 x_4 \\ x_1 x_4^3 + x_1^3 x_4 & x_2 x_4^3 + x_2^3 x_4 & x_3 x_4^3 + x_3^3 x_4 & x_4^4 + x_1^2 x_4^2 \\ + x_1 x_2^2 x_4 + x_1 x_3^2 x_4 & + x_1^2 x_2 x_4 + x_2 x_3^2 x_4 & + x_1^2 x_3 x_4 + x_2^2 x_3 x_4 & + x_2^2 x_4^2 + x_3^2 x_4^2 \end{array} \right) \mathbf{q} \mathbf{q}^T$$

$$= \frac{1}{2} \left( \begin{array}{cccc} 3\sigma_1^4 + \sigma_1^2 \sigma_2^2 + 2\sigma_{12}^2 & 3\sigma_1^2 \sigma_{12} + 3\sigma_2^2 \sigma_{12} & 3\sigma_1^2 \sigma_{13} + 3\sigma_3^2 \sigma_{13} & 3\sigma_1^2 \sigma_{14} + 3\sigma_4^2 \sigma_{14} \\ + \sigma_1^2 \sigma_3^2 + 2\sigma_{13}^2 & + \sigma_3^2 \sigma_{12} + 2\sigma_{13} \sigma_{23} & + \sigma_2^2 \sigma_{13} + 2\sigma_{12} \sigma_{23} & + \sigma_2^2 \sigma_{14} + 2\sigma_{12} \sigma_{24} \\ + \sigma_1^2 \sigma_4^2 + 2\sigma_{14}^2 & + \sigma_4^2 \sigma_{12} + 2\sigma_{14} \sigma_{24} & + \sigma_4^2 \sigma_{13} + 2\sigma_{14} \sigma_{34} & + \sigma_4^2 \sigma_{14} + 2\sigma_{14} \sigma_{44} \\ 3\sigma_1^2 \sigma_{12} + 3\sigma_2^2 \sigma_{12} & \sigma_1^2 \sigma_2^2 + 2\sigma_{12}^2 + 3\sigma_2^4 & \sigma_1^2 \sigma_{23} + 2\sigma_{12} \sigma_{13} & \sigma_1^2 \sigma_{24} + 2\sigma_{12} \sigma_{14} \\ + \sigma_3^2 \sigma_{12} + 2\sigma_{13} \sigma_{23} & + \sigma_2^2 \sigma_3^2 + 2\sigma_{23}^2 & + 3\sigma_2^2 \sigma_{23} + 3\sigma_3^2 \sigma_{23} & + 3\sigma_2^2 \sigma_{24} + 3\sigma_4^2 \sigma_{24} \\ + \sigma_4^2 \sigma_{12} + 2\sigma_{14} \sigma_{24} & + \sigma_2^2 \sigma_4^2 + 2\sigma_{24}^2 & + \sigma_4^2 \sigma_{23} + 2\sigma_{24} \sigma_{34} & + \sigma_3^2 \sigma_{24} + 2\sigma_{23} \sigma_{34} \\ 3\sigma_1^2 \sigma_{13} + 3\sigma_3^2 \sigma_{13} & \sigma_1^2 \sigma_{23} + 2\sigma_{12} \sigma_{13} & \sigma_1^2 \sigma_3^2 + 2\sigma_{13}^2 + 3\sigma_3^4 & \sigma_1^2 \sigma_{34} + 2\sigma_{13} \sigma_{14} \\ + \sigma_2^2 \sigma_{13} + 2\sigma_{12} \sigma_{23} & + 3\sigma_2^2 \sigma_{23} + 3\sigma_3^2 \sigma_{23} & + \sigma_2^2 \sigma_3^2 + 2\sigma_{23}^2 & + \sigma_2^2 \sigma_{34} + 2\sigma_{23} \sigma_{24} \\ + \sigma_4^2 \sigma_{13} + 2\sigma_{14} \sigma_{34} & + \sigma_4^2 \sigma_{23} + 2\sigma_{24} \sigma_{34} & + \sigma_3^2 \sigma_4^2 + 2\sigma_{34}^2 & + 3\sigma_3^2 \sigma_{34} + 3\sigma_4^2 \sigma_{34} \\ 3\sigma_1^2 \sigma_{14} + 3\sigma_4^2 \sigma_{14} & \sigma_1^2 \sigma_{24} + 2\sigma_{12} \sigma_{14} & \sigma_1^2 \sigma_{34} + 2\sigma_{13} \sigma_{14} & \sigma_1^2 \sigma_4^2 + 2\sigma_{14}^2 + 3\sigma_4^4 \\ + \sigma_2^2 \sigma_{14} + 2\sigma_{12} \sigma_{24} & + 3\sigma_2^2 \sigma_{24} + 3\sigma_4^2 \sigma_{24} & + \sigma_2^2 \sigma_{34} + 2\sigma_{23} \sigma_{24} & + \sigma_2^2 \sigma_4^2 + 2\sigma_{24}^2 \\ + \sigma_4^2 \sigma_{13} + 2\sigma_{14} \sigma_{34} & + \sigma_3^2 \sigma_{24} + 2\sigma_{23} \sigma_{34} & + 3\sigma_3^2 \sigma_{34} + 3\sigma_4^2 \sigma_{34} & + \sigma_3^2 \sigma_4^2 + 2\sigma_{34}^2 \end{array} \right) \mathbf{q} \mathbf{q}^T$$

3.  $\frac{1}{2}E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\}^T$
4.  $-\frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^3\Delta\check{\mathbf{q}}\mathbf{q}^T\}$

$$= -\frac{3}{2}E \left( \begin{array}{cccc} \begin{array}{l} q_1^4x_1^4 + q_1q_3^2x_1x_2^2 + 3q_1^2q_2^2x_1^2x_2^2 \\ + 3q_1^2q_2x_1^3x_2 + q_1q_3^3x_1x_3^3 \\ + 3q_1^2q_3^2x_1^2x_3^2 + 3q_1q_2q_3^2x_1x_2x_3^2 \\ + 3q_1^2q_3x_1^3x_3 + 3q_1q_2^2q_3x_1x_2^2x_3 \\ + 6q_1^2q_2q_3x_1^2x_2x_3 + q_1q_4^3x_1x_4^3 \\ + 3q_1^2q_4^2x_1^2x_4^2 + 3q_1q_2q_4^2x_1x_2x_4^2 \\ + 3q_1q_3q_4^2x_1x_3x_4^2 + 3q_1^3q_4x_1^3x_4 \\ + 3q_1^2q_2q_4x_1x_2x_4 + 6q_1^2q_2q_4x_1^2x_2x_4 \\ + 3q_1q_3^2q_4x_1x_3^2x_4 + 6q_1^2q_3q_4x_1^2x_3x_4 \\ + 6q_1q_2q_3q_4x_1x_2x_3x_4 \\ q_1^3q_2x_1^3 + q_1^4x_1x_2^3 + 3q_1^2q_2^2x_1^2x_2^2 \\ + 3q_1^2q_2^3x_1^2x_2^3 + q_2q_3^3x_1x_3^3 \\ + 3q_1q_2q_3^2x_1^2x_3^2 + 3q_2^2q_3^2x_1x_2x_3^2 \\ + 3q_1^2q_2q_3x_1^2x_3^2 + 3q_2^2q_3x_1x_2^2x_3 \\ + 6q_1q_2^2q_3x_1^2x_2x_3 + q_2q_4^3x_1x_4^3 \\ + 3q_1q_2q_4^2x_1^2x_4^2 + 3q_2^2q_4^2x_1x_2x_4^2 \\ + 3q_2q_3q_4^2x_1x_3x_4^2 + 3q_1^2q_2q_4x_1^2x_2x_4 \\ + 3q_2^2q_4x_1x_2^2x_4 + 6q_1q_2^2q_4x_1^2x_2x_4 \\ + 3q_2q_3^2q_4x_1x_3^2x_4 + 6q_1q_2q_3q_4x_1^2x_3x_4 \\ + 6q_2^2q_3q_4x_1x_2x_3x_4 \\ q_1^3q_3x_1^3 + q_2^3q_3x_1x_2^3 + 3q_1q_2^2q_3x_1^2x_2^2 \\ + 3q_1^2q_2q_3x_1^2x_2^3 + q_4^3x_1x_3^3 \\ + 3q_1q_3^2x_1^2x_3^2 + 3q_2q_3^2x_1x_2x_3^2 \\ + 3q_1^2q_3^2x_1^2x_3^2 + 3q_2^2q_3^2x_1x_2^2x_3 \\ + 6q_1q_2q_3^2x_1^2x_2x_3 + q_3q_4^3x_1x_4^3 \\ + 3q_1q_3q_4^2x_1^2x_4^2 + 3q_2q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_1x_3x_4^2 + 3q_1^2q_3q_4x_1^2x_3x_4 \\ + 3q_2^2q_3q_4x_1x_2^2x_4 + 6q_1q_2q_3q_4x_1^2x_2x_4 \\ + 3q_3^2q_4x_1x_3^2x_4 + 6q_1q_3^2q_4x_1^2x_3x_4 \\ + 6q_2q_3^2q_4x_1x_2x_3x_4 \\ q_1^3q_4x_1^3 + q_2^3q_4x_1x_2^3 + 3q_1q_2^2q_4x_1^2x_2^2 \\ + 3q_1^2q_2q_4x_1^2x_2^3 + q_3^3q_4x_1x_3^3 \\ + 3q_1q_3^2q_4x_1^2x_3^2 + 3q_2q_3^2q_4x_1x_2x_3^2 \\ + 3q_1^2q_3^2q_4x_1^2x_3^2 + 3q_2^2q_3^2q_4x_1x_2^2x_3 \\ + 6q_1q_2q_3q_4x_1^2x_2x_3 + q_4^4x_1x_4^4 \\ + 3q_1q_4^3x_1^2x_4^3 + 3q_2q_4^3x_1x_2x_4^3 \\ + 3q_3q_4^3x_1x_3x_4^3 + 3q_1^2q_4^2x_1^2x_4^2 \\ + 3q_2^2q_4^2x_1x_2^2x_4^2 + 6q_1q_2q_4^2x_1^2x_2x_4^2 \\ + 3q_3^2q_4^2x_1x_3^2x_4^2 + 6q_1q_3q_4^2x_1^2x_3x_4^2 \\ + 6q_2q_3q_4^2x_1x_2x_3x_4^2 \end{array} & \begin{array}{l} q_1^3q_2x_1^3 + q_2^4x_1x_2^4 + 3q_1q_2^3x_1^2x_2^3 \\ + 3q_1^2q_2^2x_1^2x_2^2 + q_2q_3^3x_1x_3^3 \\ + 3q_1q_2q_3^2x_1^2x_3^2 + 3q_2^2q_3^2x_1x_2x_3^2 \\ + 3q_1^2q_2q_3x_1^2x_3^2 + 3q_2^2q_3x_1x_2^2x_3 \\ + 6q_1q_2^2q_3x_1^2x_2x_3 + q_2q_4^3x_1x_4^3 \\ + 3q_1q_2q_4^2x_1^2x_4^2 + 3q_2^2q_4^2x_1x_2x_4^2 \\ + 3q_2q_3q_4^2x_1x_3x_4^2 + 3q_1^2q_2q_4x_1^2x_2x_4 \\ + 3q_2^2q_4x_1x_2^2x_4 + 6q_1q_2^2q_4x_1^2x_2x_4 \\ + 3q_2q_3^2q_4x_1x_3^2x_4 + 6q_1q_2q_3q_4x_1^2x_3x_4 \\ + 6q_2^2q_3q_4x_1x_2x_3x_4 \\ q_2^4x_2^4 + 3q_1q_2^3x_1x_2^3 + 3q_1^2q_2^2x_1^2x_2^2 \\ + q_1^3q_2x_1^3x_2 + q_2q_3^3x_2x_3^3 \\ + 3q_2^2q_3^2x_2^2x_3^2 + 3q_1q_2q_3^2x_1x_2x_3^2 \\ + 3q_2^2q_3x_2^2x_3 + 6q_1q_2^2q_3x_1x_2^2x_3 \\ + 3q_1^2q_2q_3x_1^2x_2x_3 + q_2q_4^3x_2x_4^3 \\ + 3q_2^2q_4^2x_2^2x_4^2 + 3q_1q_2q_4^2x_1x_2x_4^2 \\ + 3q_2q_3q_4^2x_2x_3x_4^2 + 3q_1^2q_2q_4x_1^2x_2x_4 \\ + 3q_2^2q_4x_2x_2^2x_4 + 6q_1q_2^2q_4x_1^2x_2x_4 \\ + 3q_2q_3^2q_4x_2x_3^2x_4 + 6q_2^2q_3q_4x_2^2x_3x_4 \\ + 6q_1q_2q_3q_4x_1x_2x_3x_4 \\ q_2^3q_3x_2^3 + 3q_1q_2^2q_3x_1x_2^2x_3 \\ + q_1^3q_3x_2^3x_3 + q_3^3x_2x_3^3 \\ + 3q_2q_3^2x_2^2x_3^2 + 3q_1q_3^2x_1x_2x_3^2 \\ + 3q_2^2q_3^2x_2^2x_3 + 6q_1q_2q_3^2x_1x_2^2x_3 \\ + 3q_1^2q_2q_3x_1^2x_2x_3 + q_3q_4^3x_2x_4^3 \\ + 3q_2q_3q_4^2x_2^2x_4^2 + 3q_1q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_2x_3x_4^2 + 3q_2^2q_3q_4x_2^2x_3x_4 \\ + 6q_1q_2q_3q_4x_1x_2x_3x_4 \\ q_2^3q_4x_2^3 + 3q_1q_2^2q_4x_1x_2^2x_4 \\ + q_1^3q_2x_1^3x_2 + q_3^3q_4x_2x_3^3 \\ + 3q_2q_3^2q_4x_2^2x_3^2 + 3q_1q_3^2q_4x_1x_2x_3^2 \\ + 3q_2^2q_3q_4x_2^2x_3 + 6q_1q_2q_3q_4x_1x_2^2x_3 \\ + 3q_1^2q_2q_3q_4x_1^2x_2x_3 + q_4^4x_2x_4^4 \\ + 3q_1q_4^3x_2^2x_4^3 + 3q_2q_4^3x_2x_2^2x_4^3 \\ + 3q_3q_4^3x_2x_3x_4^3 + 3q_1^2q_4^2x_2^2x_4^2 \\ + 3q_2^2q_4^2x_2x_2^2x_4^2 + 6q_1q_2q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_2x_3^2x_4^2 + 6q_1q_3q_4^2x_1x_2x_3x_4^2 \\ + 6q_2q_3q_4^2x_2x_2^2x_3x_4^2 \end{array} & \begin{array}{l} q_1^3q_3x_1^3 + q_2^3q_3x_1x_2^3 + 3q_1q_2^2q_3x_1^2x_2^2 \\ + 3q_1^2q_2q_3x_1^2x_2^3 + q_3^4x_1x_3^4 \\ + 3q_1q_3^2x_1^2x_3^2 + 3q_2q_3^2x_1x_2x_3^2 \\ + 3q_1^2q_3^2x_1^2x_3^2 + 3q_2^2q_3^2x_1x_2^2x_3 \\ + 6q_1q_2q_3^2x_1^2x_2x_3 + q_3q_4^3x_1x_4^3 \\ + 3q_1q_3q_4^2x_1^2x_4^2 + 3q_2q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_1x_3x_4^2 + 6q_1q_3^2q_4x_1^2x_3x_4 \\ + 6q_2q_3^2q_4x_1x_2x_3x_4 \\ q_2^3q_3x_2^3 + 3q_1q_2^2q_3x_1x_2^2x_3 \\ + q_1^3q_3x_2^3x_3 + q_3^3x_2x_3^3 \\ + 3q_2q_3^2x_2^2x_3^2 + 3q_1q_3^2x_1x_2x_3^2 \\ + 3q_2^2q_3^2x_2^2x_3 + 6q_1q_2q_3^2x_1x_2^2x_3 \\ + 3q_1^2q_2q_3x_1^2x_2x_3 + q_3q_4^3x_2x_4^3 \\ + 3q_2q_3q_4^2x_2^2x_4^2 + 3q_1q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_2x_3x_4^2 + 6q_1q_3^2q_4x_2^2x_3x_4 \\ + 6q_2q_3^2q_4x_2x_2^2x_3x_4 \\ q_3^3q_3x_3^3 + 3q_1q_2^2q_3x_1x_2^2x_3 \\ + q_1^3q_3x_3^3x_3 + q_3^3x_2x_3^3 \\ + 3q_2q_3^2x_2^2x_3^2 + 3q_1q_3^2x_1x_2x_3^2 \\ + 3q_2^2q_3^2x_2^2x_3 + 6q_1q_2q_3^2x_1x_2^2x_3 \\ + 3q_1^2q_2q_3x_1^2x_2x_3 + q_3q_4^3x_2x_4^3 \\ + 3q_2q_3q_4^2x_2^2x_4^2 + 3q_1q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_2x_3x_4^2 + 6q_1q_3^2q_4x_2^2x_3x_4 \\ + 6q_2q_3^2q_4x_2x_2^2x_3x_4 \\ q_3^3q_4x_3^3 + 3q_1q_2^2q_4x_1x_2^2x_4 \\ + q_1^3q_3q_4x_3^3x_4 + 3q_2q_3^2q_4x_2^2x_3^2 \\ + 3q_1q_3^2q_4x_2^2x_3 + 6q_1q_2q_3^2q_4x_1x_2^2x_3 \\ + 3q_1^2q_2q_3q_4x_1^2x_2x_3 + q_4^4x_3x_4^4 \\ + 3q_1q_4^3x_3^2x_4^3 + 3q_2q_4^3x_2x_2^2x_4^3 \\ + 3q_3q_4^3x_2x_3x_4^3 + 3q_1^2q_4^2x_3^2x_4^2 \\ + 3q_2^2q_4^2x_2x_2^2x_4^2 + 6q_1q_2q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_2x_3^2x_4^2 + 6q_1q_3q_4^2x_1x_2x_3x_4^2 \\ + 6q_2q_3q_4^2x_2x_2^2x_3x_4^2 \end{array} \end{array} \right)$$



$$\begin{aligned}
& 6. \frac{1}{4}E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})^2\mathbf{q}\mathbf{q}^T\}: \\
& = \frac{1}{4}E\{x_1^4 + x_2^4 + 2x_1^2x_2^2 + x_3^4 + 2x_1^2x_3^2 + 2x_2^2x_3^2 + x_4^4 + 2x_1^2x_4^2 + 2x_2^2x_4^2 + 2x_3^2x_4^2\}\mathbf{q}\mathbf{q}^T \\
& = \frac{1}{4}[3(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) + 2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2 + \sigma_1^2\sigma_3^2 + 2\sigma_{13}^2 + \sigma_1^2\sigma_4^2 + 2\sigma_{14}^2 + \sigma_2^2\sigma_3^2 + 2\sigma_{23}^2 + \sigma_2^2\sigma_4^2 + 2\sigma_{24}^2 + \sigma_3^2\sigma_4^2 + 2\sigma_{34}^2)]\mathbf{q}\mathbf{q}^T
\end{aligned}$$

$$\begin{aligned}
& 7. -\frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\mathbf{q}\mathbf{q}^T\} \\
& = -\frac{3}{2}E\left\{\begin{aligned} & \left( \begin{aligned} & q_1^2x_1^4 + q_2^2x_2^4 + 2q_1q_2x_1x_2^3 + q_1^2x_1^2x_2^2 + q_2^2x_1^2x_2^2 \\ & + 2q_1q_2x_1^3x_2 + q_3^2x_3^4 + 2q_1q_3x_1x_3^3 + 2q_2q_3x_2x_3^3 + q_1^2x_1^2x_3^2 \\ & + q_3^2x_1^2x_3^2 + q_2^2x_2^2x_3^2 + q_3^2x_2^2x_3^2 + 2q_1q_2x_1x_2x_3^2 + 2q_1q_3x_1^3x_3 \\ & + 2q_2q_3x_2^3x_3 + 2q_1q_3x_1x_2^2x_3 + 2q_2q_3x_1^2x_2x_3 + q_4^2x_4^4 + 2q_1q_4x_1x_4^3 \\ & + 2q_2q_4x_2x_4^3 + 2q_3q_4x_3x_4^3 + q_1^2x_1^2x_4^2 + q_4^2x_1^2x_4^2 + q_2^2x_2^2x_4^2 \\ & + q_4^2x_2^2x_4^2 + 2q_1q_2x_1x_2x_4^2 + q_3^2x_3^2x_4^2 + q_4^2x_3^2x_4^2 \\ & + 2q_1q_3x_1x_3x_4^2 + 2q_2q_3x_2x_3x_4^2 + 2q_1q_4x_1^3x_4 + 2q_2q_4x_2^3x_4 \\ & + 2q_1q_4x_1x_2^2x_4 + 2q_2q_4x_1^2x_2x_4 + 2q_3q_4x_3^3x_4 + 2q_1q_4x_1x_3^2x_4 \\ & + 2q_2q_4x_2x_3^2x_4 + 2q_3q_4x_1^2x_3x_4 + 2q_3q_4x_2^2x_3x_4 \end{aligned} \right) \mathbf{q}\mathbf{q}^T \\ & \left( \begin{aligned} & 3q_1^2\sigma_1^4 + q_1^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2 + \sigma_1^2\sigma_3^2 + 2\sigma_{13}^2 + \sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\ & 3q_2^2\sigma_2^4 + q_2^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2 + \sigma_2^2\sigma_3^2 + 2\sigma_{23}^2 + \sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\ & 6q_1q_2(\sigma_2^2\sigma_{12} + \sigma_1^2\sigma_{12}) + 2q_1q_2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23} + \sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\ & 3q_3^2\sigma_3^4 + q_3^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2 + \sigma_2^2\sigma_3^2 + 2\sigma_{23}^2 + \sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\ & 6q_1q_3(\sigma_3^2\sigma_{13} + \sigma_1^2\sigma_{13}) + 2q_1q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23} + \sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\ & 6q_2q_3(\sigma_3^2\sigma_{23} + \sigma_2^2\sigma_{23}) + 2q_2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13} + \sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ & 3q_4^2\sigma_4^4 + q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2 + \sigma_2^2\sigma_4^2 + 2\sigma_{24}^2 + \sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\ & 6q_1q_4(\sigma_4^2\sigma_{14} + \sigma_1^2\sigma_{14}) + 2q_1q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24} + \sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ & 6q_2q_4(\sigma_4^2\sigma_{24} + \sigma_2^2\sigma_{24}) + 2q_2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14} + \sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\ & 6q_3q_4(\sigma_4^2\sigma_{34} + \sigma_3^2\sigma_{34}) + 2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14} + \sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \end{aligned} \right) \mathbf{q}\mathbf{q}^T \end{aligned} \right\}
\end{aligned}$$

$$\begin{aligned}
& 8. \frac{9}{4}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^4\mathbf{q}\mathbf{q}^T\} \\
& = \frac{9}{4}E\left\{\begin{aligned} & \left( \begin{aligned} & q_1^4x_1^4 + q_2^4x_2^4 + 4q_1q_2^3x_1x_2^3 + 6q_1^2q_2^2x_1^2x_2^2 \\ & + 4q_1^3q_2x_1^3x_2 + q_3^4x_3^4 + 4q_1q_3^3x_1x_3^3 + 4q_2q_3^3x_2x_3^3 \\ & + 6q_1^2q_3^2x_1^2x_3^2 + 6q_2^2q_3^2x_2^2x_3^2 + 12q_1q_2q_3^2x_1x_2x_3^2 \\ & + 4q_1^3q_3x_1^3x_3 + 4q_2^3q_3x_2^3x_3 + 12q_1q_2^2q_3x_1x_2^2x_3 \\ & + 12q_1^2q_2q_3x_1^2x_2x_3 + q_4^4x_4^4 + 4q_1q_4^3x_1x_4^3 + 4q_2q_4^3x_2x_4^3 \\ & + 4q_3q_4^3x_3x_4^3 + 6q_1^2q_4^2x_1^2x_4^2 + 6q_2^2q_4^2x_2^2x_4^2 \\ & + 12q_1q_1q_2q_4^2x_1x_2x_4^2 + 6q_3 + 6q_3^2q_4^2x_3^2x_4^2 \\ & + 12q_1q_3q_4^2x_1x_3x_4^2 + 12q_2q_3q_4^2x_2x_3x_4^2 + 4q_1^3q_4x_1^3x_4 \\ & + 4q_2^3q_4x_2^3x_4 + 12q_1q_2^2q_4x_1x_2^2x_4 + 12q_1^2q_2q_4x_1^2x_2x_4 \\ & + 4q_3^3q_4x_3^3x_4 + 12q_3q_3^2q_4x_1x_3^2x_4 + 12q_2q_3^2q_4x_2x_3^2x_4 \\ & + 12q_1^2q_3q_4x_1^2x_3x_4 + 12q_2^2q_3q_4x_2^2x_3x_4 \\ & + 24q_1q_2q_3q_4x_1x_2x_3x_4 \end{aligned} \right) \mathbf{q}\mathbf{q}^T \\ & \left( \begin{aligned} & 3q_1^4\sigma_1^4 + 3q_2^4\sigma_2^4 + 12q_1q_3^2\sigma_2^2\sigma_{12} + 6q_1^2q_2^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) + 12q_1^3q_2\sigma_1^2\sigma_{12} + 3q_3^4\sigma_3^4 + 12q_1q_3^2\sigma_3^2\sigma_{13} \\ & 12q_2q_3^2\sigma_3^2\sigma_{23} + 6q_1^2q_3^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) + 6q_2^2q_3^2(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) + 12q_1q_2q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) + 12q_1^3q_3\sigma_1^2\sigma_{13} \\ & 12q_2^3q_3\sigma_2^2\sigma_{23} + 12q_1q_2^2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) + 12q_1^2q_2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) + 3q_4^4\sigma_4^4 + 12q_1q_4^3\sigma_4^2\sigma_{14} \\ & 12q_2q_4^3\sigma_4^2\sigma_{24} + 12q_3q_4^3\sigma_4^2\sigma_{34} + 6q_1^2q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) + 6q_2^2q_4^2(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) + 12q_1q_2q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\ & 6q_3^2q_4^2(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) + 12q_1q_3q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) + 12q_2q_3q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) + 12q_1^3q_4\sigma_1^2\sigma_{14} \\ & 12q_2^3q_4\sigma_2^2\sigma_{24} + 12q_1q_2^2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) + 12q_1^2q_2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) + 12q_3^3q_4\sigma_3^2\sigma_{34} + 12q_1q_3^2q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ & 12q_2q_3^2q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) + 12q_1^2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) + 12q_2^2q_3q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) + 24(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{aligned} \right) \mathbf{q}\mathbf{q}^T \end{aligned} \right\}
\end{aligned}$$